REDUCED SCHUR FUNCTIONS AND THE LITTLEWOOD-RICHARDSON COEFFICIENTS

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§0 Introduction

The present paper deals with a formula satisfied by "r-reduced" Schur functions.

Schur functions originally appear as irreducible characters of general linear group over the complex number field. In this paper they are considered as weighted homogeneous polynomials with respect to the power sum symmetric functions. More precisely, for a Young diagram λ of size n, the Schur function indexed by λ reads

$$S_\lambda(t) = \sum_{\nu_1+2\nu_2+\cdots=n} \chi^\lambda(\nu) \frac{t_1^{\nu_1} t_2^{\nu_2} \cdots}{\nu_1! \nu_2! \cdots},$$

where $\chi^\lambda(\nu)$ is the character value of the irreducible representation S^λ of the group algebra $\mathbb{Q}\mathfrak{S}_n$, evaluated at the conjugacy class of the cycle type $\nu = (1^{\nu_1} 2^{\nu_2} \cdots n^{\nu_n})$. Setting $t_{jr} = 0$ for $j = 1, 2, \ldots$ in $S_\lambda(t)$, we have the r-reduced Schur function $S_\lambda^{(r)}(t)$. The set of all r-reduced Schur functions spans the polynomial ring $P^{(r)} = \mathbb{Q}[t_j; j \not\equiv 0 (\mathrm{mod}\ r)]$. We show that a good choice of basis elements leads to an explicit description of all other r-reduced Schur functions involving the Littlewood-Richardson coefficients.

The formula has not only a purely combinatorial meaning, but also nice implications in two different fields. One is about the basic representation of the affine Lie algebra $A_{r-1}^{(1)}$. We show that the basis in the main theorem gives in turn a weight basis of the basic $A_{r-1}^{(1)}$-module realized in $P^{(r)}$. The other is about modular representations of the symmetric group. Our explicit formula implies that the determination of the decomposition matrices reduces to that for the basic set we give in this paper.

The paper is organized as follows. In Section 1 we introduce generalized Maya diagrams and associated r-reduced Schur functions. Section 2 is for a combinatorics of Young diagrams. Section 3 is devoted to the main theorem. In Section 4 we describe weight

vectors of the basic $A_{r-1}^{(1)}$-module. In Section 5 the formula is translated into that in the modular representation theory.

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§1 Generalized Maya Diagrams and Schur Functions

In this section we will summarize known results on Schur functions.

Let $\Lambda_n = \mathbb{Q}[x_1, x_2, \dots, x_n]^{\mathfrak{S}_n}$ be the ring of symmetric polynomials of n variables. For $m < n$ we have a surjective homomorphism $P_{nm} : \Lambda_n \longrightarrow \Lambda_m$ by setting $x_{m+1} = x_{m+2} = \cdots = x_n = 0$. In this way we obtain an inverse system and denote $\Lambda = \varprojlim \Lambda_n$. The power sum symmetric function $p_j(x)$ is by definition $p_j(x) = \varprojlim p_j(x_1, \dots, x_n)$, where $p_j(x_1, \dots, x_n) = x_1^j + \cdots + x_n^j$. It is known that Λ is generated by $p_j(x)$ $(j = 1, 2, \dots)$ and these elements are algebraically independent.

Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a semi-infinite sequence with integer components satisfying $\alpha_{i+1} = \alpha_i - 1$ for sufficiently large i. We shall introduce an equivalence relation on the set of such sequences as follows: $\alpha \sim \beta$ if $\alpha_i - \beta_i$ is independent of i. The quotient by the equivalence relation is denoted by

$$\mathcal{M} = \{\alpha = (\alpha_1, \alpha_2, \dots) | \alpha_{i+1} = \alpha_i - 1 \text{ for all } i \gg 0\} / \sim$$

and an element of $\mathcal{M}$ is called a generalized Maya diagram. We let $\mathcal{M}_d$ be a subset of $\mathcal{M}$ consisting of strictly decreasing sequences.

We now define Schur functions indexed by generalized Maya diagrams. For each $\alpha \in \mathcal{M}$, we put

$$S_\alpha(x) = \varprojlim S_\alpha(x_1, \dots, x_n),$$

where

$$S_\alpha(x_1,\ldots,x_n)=\frac{\det\left(x_i^{\alpha_j-\alpha_n}\right)_{1\le i,j\le n}}{\det\left(x_i^{-j+n}\right)_{1\le i,j\le n}}$$

for a sufficiently large n. We can easily see that $S_\alpha(x_1,\ldots,x_n)$ are well-defined and that $P_{nm}(S_\alpha(x_1,\ldots,x_n))=S_\alpha(x_1,\ldots,x_m)$ for $m<n$.

Let $\mathfrak{S}_n$ be the symmetric group of degree n and put

$$\mathfrak{S}_\infty=\bigcup_{n\ge 1}\mathfrak{S}_n.$$

By permuting the first n components, $\mathfrak{S}_n$ acts on $\mathcal{M}$. If there exists a unique element $\sigma\in\mathfrak{S}_\infty$ such that $\sigma\alpha\in\mathcal{M}_d$, the sign of σ is referred to as the sign of α and denoted by $\mathrm{sgn}(\alpha)$. Otherwise we put $\mathrm{sgn}(\alpha)=0$.

For $\alpha\in\mathcal{M}$ we define a sequence $\lambda(\alpha)=(\lambda_1,\lambda_2,\ldots)$ by putting $\lambda_j=\alpha_j-\alpha_n+(j-n)$ for a sufficiently large n. The size $|\lambda(\alpha)|$ is defined by $\sum\lambda_j$. We will write $|\alpha|$ instead of $|\lambda(\alpha)|$ for $\alpha\in\mathcal{M}$. Note that in case of $\alpha\in\mathcal{M}_d$, $\lambda(\alpha)$ is a Young diagram. We often identify an element of $\mathcal{M}_d$ with a Young diagram. The transposed Young diagram of $\lambda(\alpha)$ is denoted by $\lambda(\alpha)'$ and the corresponding element in $\mathcal{M}_d$ is denoted by α', i.e., $\lambda(\alpha)'=\lambda(\alpha')$. For $\alpha,\beta_1,\ldots,\beta_m\in\mathcal{M}$, the Littlewood-Richardson coefficients are defined by

$$C^{\alpha}_{\beta_1\cdots\beta_m}=\begin{cases} LR^{\lambda(\alpha)}_{\lambda(\beta_1)\cdots\lambda(\beta_m)} & \text{if } \exists\sigma,\ \tau_j\in\mathfrak{S}_\infty\ (j=1,\ldots,m) \text{ such that } \sigma\alpha,\ \tau_j\beta_j\in\mathcal{M}_d\\ 0 & \text{otherwise,}\end{cases}$$

where

$$S_{\lambda_1}(x)\cdots S_{\lambda_m}(x)=\sum_\mu LR^{\mu}_{\lambda_1\cdots\lambda_m}S_\mu(x).$$

The following properties of the Schur functions are well known [9].

Proposition 1.1 *(1)* $S_{\sigma\alpha}(x) = \mathrm{sgn}(\sigma) S_\alpha(x)$ *for any* $\sigma \in \mathfrak{S}_\infty$.

(2) For any $j \geq 1$ *and* $\alpha \in \mathcal{M}$, *the sequence* $\alpha + l\epsilon_i$ *is in* $\mathcal{M}$ *and*

$$S_\alpha(x) p_j(x) = \sum_{i \geq 1} S_{\alpha + j\epsilon_i}(x),$$

where $\epsilon_i = (0, \ldots, 0, \overset{i}{\breve{1}}, 0, \ldots)$.

(3) The set $\{S_\alpha(x) | \alpha \in \mathcal{M}_d\}$ *gives a basis of* Λ.

(4) For $\alpha \in \mathcal{M}_d$,

$$S_\alpha(x, y) = \sum_{\beta, \gamma \in \mathcal{M}_d} C^{\alpha}_{\beta\gamma} S_\beta(x) S_\gamma(y).$$

(5) Let $\omega : \Lambda \longrightarrow \Lambda$ *be an algebra automorphism defined by* $\omega(p_j(x)) = (-1)^{j-1} p_j(x)$. *Then*

$$\omega\left(S_\alpha(x)\right) = S_{\alpha'}(x) \ \ (\alpha \in \mathcal{M}_d).$$

Note that the summation in (2) is a finite sum. If $\alpha \notin M_d$, then we define $S_{\alpha'}(x)$ by $S_{\alpha'}(x) = \omega(S_\alpha(x))$. This definition is compatible with above (1) and (5).

Let $\rho^{(r)} : \Lambda = \mathbb{Q}[p_1, p_2, \ldots] \longrightarrow P^{(r)} = \mathbb{Q}[t_j ; j \not\equiv 0 \pmod r]$ be the surjective homomorphism of algebras defined by

$$\begin{cases} p_j(x) & \longmapsto & jt_j & \text{if } j \not\equiv 0 \pmod r \\ p_j(x) & \longmapsto & 0 & \text{otherwise.} \end{cases}$$

We shall write $S^{(r)}_\alpha(t) = \rho^{(r)}\left(S_\alpha(x)\right)$ and call it the r-reduced Schur function indexed by α. When $\alpha \in \mathcal{M}_d$ we may write $S^{(r)}_{\lambda(\alpha)}(t)$ instead of $S^{(r)}_\alpha(t)$. The following are easily checked by using Proposition 1.1(1) and (2).

Lemma 1.2 *(1)* $S^{(r)}_{\sigma\alpha}(t) = \mathrm{sgn}(\sigma) S^{(r)}_{\alpha}(t)$. *(2)* $\displaystyle\sum_{i\geq 1} S^{(r)}_{\alpha+jr\epsilon_i}(t) = 0 \ (j \geq 1)$.

§2 Cores and Quotients

Let $\alpha \in \mathcal{M}$ be a generalized Maya diagram. Define, for $0 \leq k \leq r-1$,

$$\alpha^{(k)} = \{\alpha_i | \alpha_i \equiv k (\mathrm{mod}\ r)\}, \alpha^{(k)}_{+} = \left\{\alpha_i \in \alpha^{(k)} | \alpha_i \geq 0\right\}, \alpha^{(k)}_{-} = \left\{\alpha_i \in \alpha^{(k)} | \alpha_i < 0\right\}.$$

We fix a sequence in the equivalence class α satisfying the following condition: (1) $\displaystyle\bigcup_k \alpha^{(k)}_{-} = \{-1, -2, \ldots\}$, (2) cardinality $\sum_k n_k$ is divisible by r, where $n_k = \#\alpha^{(k)}_{+}$.

We can form new elements $\alpha[k] \in \mathcal{M}$ and $\alpha_c \in \mathcal{M}_d$ as follows. The sequence $\alpha[k]$ is obtained by

$$\alpha[k] = \left(\frac{\alpha^{(k)}_{i_1} - k}{r}, \frac{\alpha^{(k)}_{i_2} - k}{r}, \ldots \right),$$

where $(\alpha^{(k)}_{i_1}, \alpha^{(k)}_{i_2}, \ldots)$ is the subsequence consisting of elements of $\alpha^{(k)}$. We have α_c if we replace $(\alpha^{(k)}_{i_1}, \alpha^{(k)}_{i_2}, \ldots)$ by $(n_k r + k, (n_k - 1)r + k, \ldots, k, -r + k, \ldots)$.

In the case $\alpha \in \mathcal{M}_d$, $\lambda(\alpha_c)$ and $(\lambda(\alpha[0]), \ldots, \lambda(\alpha[r-1]))$ is called the r-core and the r-quotient of $\lambda(\alpha)$, respectively [11]. The r-core is obtained from λ by removing r-hooks successively as many as possible.

Since non-negative entries of $(\alpha^{(0)}_1, \ldots, \alpha^{(r-1)}_1, \alpha^{(0)}_2, \ldots, \alpha^{(r-1)}_2, \ldots)$ is a permutation of that of $(\alpha_1, \alpha_2, \ldots)$, we define the r-sign $\delta_r(\alpha)$ of $\alpha \in \mathcal{M}_d$ to be the sign of this permutation. For an arbitrary $\alpha \in \mathcal{M}$, we define

$$\delta_r(\alpha) = \begin{cases} \delta_r(\sigma\alpha) & \text{if } \exists \sigma \in \mathfrak{S}_\infty \text{ such that } \sigma\alpha \in \mathcal{M}_d \\ 0 & \text{otherwise.} \end{cases}$$

If $r=2$ and $\alpha \in \mathcal{M}_d$, then $\lambda(\alpha_c)$ is a staircase Young diagram and $\delta_2(\alpha)$ is simply described as $(-1)^q$, where q is the number of column 2-hooks to be removed in the procedure of completing the 2-core of $\lambda(\alpha)$.

If $\alpha \in \mathcal{M}$ corresponds to the $(r+1)$-tuple $(\alpha_c; \alpha[0], \ldots, \alpha[r-1])$ we may also write $S^{(r)}_{(\alpha_c;\alpha[0],\ldots,\alpha[r-1])}(t)$ instead of $S^{(r)}_\alpha(t)$. We have

$$S^{(r)}_{(\alpha_c;\sigma_0\alpha[0],\ldots,\sigma_{r-1}\alpha[r-1])}(t) = \prod_{k=0}^{r-1} \mathrm{sgn}(\sigma_k) S^{(r)}_{(\alpha_c;\alpha[0],\ldots,\alpha[r-1])}(t) \qquad \text{for } \sigma_k \in \mathfrak{S}_\infty.$$

§3 Main Theorem

We are now ready to state our main theorem.

Theorem 3.1 *For any Young diagram λ we have*

$$S^{(r)}_\lambda(t) = (-1)^{|\lambda[0]|}\delta_r(\lambda) \sum_{\mu,\nu_1,\ldots,\nu_{r-1}} LR^{\lambda[0]'}_{\nu_1\cdots\nu_{r-1}} LR^{\mu[1]}_{\nu_1\lambda[1]} \cdots LR^{\mu[r-1]}_{\nu_{r-1}\lambda[r-1]} \delta_r(\mu) S^{(r)}_\mu(t),$$

where summation runs over Young diagrams μ and $\nu_1, \ldots, \nu_{r-1}$ such that $|\mu| = |\lambda|$, $\mu[0] = \emptyset$ and the core of μ coincides with that of λ.

Let θ denote the element of $\mathcal{M}_d$ corresponding to the empty Young diagram $\emptyset$. We shall show that, for any $\alpha \in \mathcal{M}$,

$$S^{(r)}_\alpha(t) = (-1)^{|\alpha[0]|}\mathrm{sgn}(\alpha)\delta_r(\alpha) \sum_{\beta,\gamma_1,\ldots,\gamma_{r-1}} C^{\alpha[0]'}_{\gamma_1\cdots\gamma_{r-1}} C^{\beta[1]}_{\gamma_1\alpha[1]} \cdots C^{\beta[r-1]}_{\gamma_{r-1}\alpha[r-1]} \delta_r(\beta) S^{(r)}_\beta(t),$$

where summation runs over $\beta, \gamma_1, \ldots, \gamma_{r-1} \in \mathcal{M}_d$ such that $|\beta| = |\alpha|$, $\beta[0] = \theta$ and $\beta_c = \alpha_c$. Let $F_\alpha(t)$ be the right hand side. We focus on identities satisfied by $F_\alpha(t)$ and $S^{(r)}_\alpha(t)$. We first need a lemma.

Lemma 3.2 *Let U be the vector space*

$$U := \bigoplus_{\alpha \in \mathcal{M}} \mathbb{Q} u_\alpha \Big/ \sum_{\alpha \in \mathcal{M}, \sigma \in \mathfrak{S}_\infty} \mathbb{Q}(u_{\sigma\alpha} - \mathrm{sgn}(\sigma) u_\alpha) .$$

Then we have the following:

(1) U has a basis $\{u_\alpha | \alpha \in M_d\}$.

(2) If we set

$$U_1 := U \Big/ \sum_{j \geq 1} \mathbb{Q} \left(\sum_{i \geq 1} u_{\alpha + j\epsilon_i} \right) ,$$

then $U_1 \simeq \mathbb{Q}$.

(3) If we set

$$U_r := U \Big/ \sum_{j \geq 1} \mathbb{Q} \left(\sum_{i \geq 1} u_{\alpha + jr\epsilon_i} \right) ,$$

then there is a linear isomorphism

$$\begin{array}{ccc} U_r & \xrightarrow{\sim} & P^{(r)} \\ \Cup & & \Cup \\ u_\alpha & \longmapsto & S_\alpha^{(r)}(t). \end{array}$$

Proof. (1) is obvious.

(2) We have a canonical linear surjection from Λ to U_1 which maps S_α to u_α. Then the kernel of this surjection coincides with the maximal ideal $\mathcal{I} = (p_1(x), p_2(x), \ldots)$ of Λ because of the relations in Proposition 1.1(2) and $\sum_{j \geq 1} u_{\alpha + j\epsilon_i} = 0$ in U_1. Since the algebra $\Lambda/\mathcal{I}$ is isomorphic to $\mathbb{Q}$, we have $U_1 \simeq \Lambda/\mathcal{I} \simeq \mathbb{Q}$.

(3) Consider a linear surjection $T : \Lambda \longrightarrow U_r$ defined by $T(S_\alpha(x)) = u_\alpha$. Put $\mathcal{I}^{(r)} = (p_r(x), p_{2r}(x), \ldots)$. Then, by using Proposition 1.1(2), we have $T(\mathcal{I}^{(r)}) = 0$ and a surjection $\bar{T} : \Lambda/\mathcal{I}^{(r)} \longrightarrow U_r$. On the other hand we can define a linear surjection $S : U_r \longrightarrow P^{(r)}$ by $S(u_\alpha) = S_\alpha^{(r)}(t)$. The composition $S \circ \bar{T}$ gives a linear isomorphism from $\Lambda/\mathcal{I}^{(r)}$ to $P^{(r)}$. Hence we have that S is a linear isomorphism as desired. ■

This lemma leads to the following proposition.

Proposition 3.3 *The set* $\left\{S_\alpha^{(r)}(t) \mid \alpha \in \mathcal{M}_d,\ \alpha[0] = \theta\right\}$ *gives a basis of* $P^{(r)}$.

Proof. We first show that these elements span $P^{(r)}$. Introduce a filtration $\{V_n\}_{n\geq 0}$ of $P^{(r)}$ by

$$V_n = \sum_{\alpha;|\alpha[0]|\leq n} \mathbb{Q}S_\alpha^{(r)}(t),$$

and put

$$\bar{V} = \bigoplus_{n\geq 0} V_n/V_{n-1},$$

where $V_{-1} = \{0\}$. The set $\{\bar{S}_\alpha^{(r)}(t) | \alpha \in \mathcal{M}_d,\ |\alpha[0]| = n\}$ spans the direct summand V_n/V_{n-1}. Here $\bar{S}_\alpha^{(r)}(t)$ stands for the modulo class represented by $S_\alpha^{(r)}(t)$. The equation in Lemma 1.2(2) reads

$$0 = \sum_{i\geq 1} S_{\alpha+jr\epsilon_i}^{(r)}(t) = \sum_{i\geq 1} S_{(\alpha_c;\alpha[0]+j\epsilon_i,\alpha[1],\ldots,\alpha[r-1])}^{(r)}(t) + \sum_{k=1}^{r-1}\sum_{i\geq 1} S_{(\alpha_c;\alpha[0],\ldots,\alpha[k]+j\epsilon_i,\ldots,\alpha[r-1])}^{(r)}(t).$$

If $|\alpha[0] + j\epsilon_i| = n$, then $\bar{S}_{(\alpha_c;\alpha[0],\ldots,\alpha[k]+j\epsilon_i,\ldots,\alpha[r-1])}^{(r)}(t) = 0$. Hence

$$\sum_{i\geq 1} \bar{S}_{(\alpha_c;\alpha[0]+j\epsilon_i,\alpha[1],\ldots,\alpha[r-1])}^{(r)}(t) = 0.$$

If we set $\bar{W} = \sum_{\beta \in \mathcal{M}} \bar{S}^{(r)}_{(\alpha_c;\beta,\alpha[1],\ldots,\alpha[r-1])}(t)$, then it turns out to be $\bar{W} = \mathbb{Q}\bar{S}_{(\alpha_c;\theta,\alpha[1],\ldots,\alpha[r-1])}(t)$ by applying Lemma 3.2(2). Thus we have the first part of the proof.

Next we will show the linear independence. We make $P^{(r)}$ into a graded algebra by putting $\deg t_j = j$. The dimension of the homogenous component of degree n is $p^{(r)}(n)$, the number of partitions of n into positive integers not divisible by r. We denote by $d^{(r)}(n)$ the cardinality of the set $\left\{S^{(r)}_\alpha(t) \mid \alpha \in \mathcal{M}_d,\ \alpha[0] = \theta, \deg S^{(r)}_\alpha(t) = n\right\}$. By the one-to-one correspondence between the set of all Young diagrams and the set of $(r+1)$-tuples of r-cores and r-quotients, we have

$$\sum_n d^{(r)}(n)q^n = \frac{\phi(q^r)}{\phi(q)} = \sum_n p^{(r)}(n)q^n,$$

where $\phi(q) = \prod_{j=1}^{\infty}(1-q^j)$. This proves the linear independence. ■

We now look at identities satisfied by

$$F_\alpha(t) = (-1)^{|\alpha[0]|}\mathrm{sgn}(\alpha)\delta_r(\alpha) \sum_{\beta,\gamma_1,\ldots,\gamma_{r-1}} C^{\alpha[0]'}_{\gamma_1\cdots\gamma_{r-1}} C^{\beta[1]}_{\gamma_1\alpha[1]} \cdots C^{\beta[r-1]}_{\gamma_{r-1}\alpha[r-1]}\delta_r(\beta)S^{(r)}_\beta(t).$$

Proposition 3.4 $\displaystyle\sum_{i\geq 1} F_{\alpha+jr\epsilon_i}(t) = 0 \ (j \geq 1).$

Proof. We shall prove, for any β,

$$\sum_{i\geq 1}(-1)^{|(\alpha+jr\epsilon_i)[0]|}\mathrm{sgn}(\alpha+jr\epsilon_i)\delta_r(\alpha+jr\epsilon_i) \sum_{\gamma_1,\ldots,\gamma_{r-1}} C^{(\alpha+jr\epsilon_i)[0]'}_{\gamma_1\cdots\gamma_{r-1}} C^{\beta[1]}_{\gamma_1(\alpha+jr\epsilon_i)[1]} \cdots C^{\beta[r-1]}_{\gamma_{r-1}(\alpha+jr\epsilon_i)[r-1]} = 0.$$

Since

$$\frac{\mathrm{sgn}(\alpha+jr\epsilon_i)\delta_r(\alpha+jr\epsilon_i)}{\mathrm{sgn}(\alpha)\delta_r(\alpha)} = \frac{\mathrm{sgn}(\alpha[k]+j\epsilon_i)}{\mathrm{sgn}(\alpha[k])}$$

for some k, it is enough to show that

$$\sum_{i\geq 1}\Big\{(-1)^j \mathrm{sgn}(\alpha[0]+j\epsilon_i) C^{(\alpha[0]+j\epsilon_i)'}_{\gamma_1\cdots\gamma_{r-1}} C^{\beta[1]}_{\gamma_1\alpha[1]}\cdots C^{\beta[r-1]}_{\gamma_{r-1}\alpha[r-1]}$$

$$+\ \sum_{k=1}^{r-1}\mathrm{sgn}(\alpha[k]+j\epsilon_i)\ C^{\alpha[0]'}_{\gamma_1\cdots\gamma_{r-1}} C^{\beta[1]}_{\gamma_1\alpha[1]}\cdots C^{\beta[k]}_{\gamma_k(\alpha[k]+j\epsilon_i)}\cdots C^{\beta[r-1]}_{\gamma_{r-1}\alpha[r-1]}\Big\} = 0.$$

Taking a summation over $\beta[1],\ldots,\beta[r-1]$ after multiplying by $\prod_{k=1}^{r-1} S_{\beta[k]}(x^{(k)})$, it reduces to

$$\sum_{i\geq 1}(-1)^j S_{(\alpha[0]+j\epsilon_i)'}(x^{(1)},\ldots,x^{(r-1)})\prod_{k=1}^{r-1} S_{\alpha[k]}(x^{(k)})$$

$$+\ \sum_{k=1}^{r-1}\sum_{i\geq 1} S_{(\alpha[0])'}(x^{(1)},\ldots,x^{(r-1)}) S_{\alpha[1]}(x^{(1)})\cdots S_{\alpha[k]+j\epsilon_i}(x^{(k)})\cdots S_{\alpha[r-1]}(x^{(r-1)}) = 0.$$

Applying Proposition 1.1(2) and (5) we see

$$(-1)^{2j-1} S_{\alpha[0]'}(x^{(1)},\ldots,x^{(r-1)}) p_j(x^{(1)},\ldots,x^{(r-1)})\prod_{k=1}^{r-1} S_{\alpha[k]}(x^{(k)})$$

$$+\ \sum_{k=1}^{r-1} S_{\alpha[0]'}(x^{(1)},\ldots,x^{(r-1)}) p_j(x^{(k)})\prod_{l=1}^{r-1} S_{\alpha[l]}(x^{(l)}) = 0. \quad \blacksquare$$

Finally we complete the proof of the main theorem. By Lemma 3.2(3) we have a linear mapping $P^{(r)} \longrightarrow P^{(r)}$, which maps $S^{(r)}_\alpha(t)$ to $F_\alpha(t)$. It turns out to be the identity since $S^{(r)}_\alpha(t)$ equals $F_\alpha(t)$ for the basis given in Proposition 3.3.

§4 Weight Vectors in the Basic $A^{(1)}_{r-1}$-Module

We can explicitly obtain a basis of each weight space of the basic $A^{(1)}_{r-1}$-module [1, 5, 6] by using the results in Section 3.

Consider the affine Lie algebra of type $A^{(1)}_{r-1}$. The Cartan subalgebra is $\oplus_{i=0}^{r-1}\mathbb{Q}\alpha_i^\vee \oplus \mathbb{Q}d_0$, where $\langle d_0, \alpha_i\rangle = 1$ for $0 \le i \le r-1$. The basic $A^{(1)}_{r-1}$-module is the simple highest weight module with highest weight Λ_0 where $\langle \alpha_i^\vee, \Lambda_0\rangle = \delta_{i0}$ and $\langle d_0, \Lambda_0\rangle = 0$.

Let W be a vector space over $\mathbb{Q}$ with a basis $\{\psi(i), \psi^*(i) \mid i \in \mathbb{Z}\}$. The Clifford algebra $\mathcal{A}$ is an associative algebra generated by W with respect to the following relations:

$$\{\psi^*(i), \psi(j)\} = \delta_{ij},$$
$$\{\psi(i), \psi(j)\} = 0 = \{\psi^*(i), \psi^*(j)\},$$

where we have put $\{a, b\} = ab + ba$. We split W into two subspaces given by $W_a = (\oplus_{i<0}\mathbb{Q}\psi(i)) \oplus (\oplus_{i\ge 0}\mathbb{Q}\psi^*(i))$ and $W_c = (\oplus_{i\ge 0}\mathbb{Q}\psi(i)) \oplus (\oplus_{i<0}\mathbb{Q}\psi^*(i))$. Then the Fock space (resp. the dual Fock space) is the left (resp. right) $\mathcal{A}$-module $\mathcal{F} = \mathcal{A}/\mathcal{A}W_a = \mathcal{A}|0\rangle$ (resp. $\mathcal{F}^* = W_c\mathcal{A}\backslash\mathcal{A} = \langle 0|\mathcal{A}$), where $|0\rangle = 1 \bmod \mathcal{A}W_a$ (resp. $\langle 0| = 1 \bmod W_c\mathcal{A}$). A linear form $\langle 0|a|0\rangle \in \mathbb{Q}$ $(a \in \mathcal{A})$ is uniquely determined by setting $\langle 0|1|0\rangle = 1$.

Define the charge zero sector $\mathcal{F}[0]$ as a subspace of $\mathcal{F}$ spanned by the elements $\Psi = \psi^*(i_1)\psi^*(i_2)\cdots\psi^*(i_N)\psi(j_1)\psi(j_2)\cdots\psi(j_N)|0\rangle$ $(j_N > \cdots > j_1 \ge 0 > i_N > \cdots > i_1)$, namely (# of $\{\psi\}$) = (# of $\{\psi^*\}$). The space $\mathcal{F}[0]$ affords an irreducible representation of the Lie algebra $\mathfrak{gl}(\infty) = \mathfrak{g} \oplus \mathbb{Q}\cdot 1$, where $\mathfrak{g} = \{\sum a_{ij} :\psi(i)\psi^*(j): \mid a_{ij} \in \mathbb{Q},\ \ a_{ij} = 0$ if $|i-j| \gg 0\}$. Note that the sum in the definition is infinite sum.

The boson-fermion correspondence gives an isomorphism $\mathcal{F}[0] \simeq \mathbb{Q}[t_1, t_2, \ldots]$ as $\mathfrak{gl}(\infty)$-modules. More precisely, the basis vector above corresponds to,up to sign, the Schur function associated with the generalized Maya diagram $(j_N, \ldots, j_1, \bar{i}_1, \bar{i}_2, \ldots)$, where $\{\bar{i}_k\}_{k\in\mathbb{Z}} = \{-1, -2, \ldots\}\backslash\{i_1, i_2, \ldots, i_N\}$ and $\bar{i}_1 > \bar{i}_2 > \cdots$.

As is shown in [8], the Chevalley generators of $A^{(1)}_{r-1}$ are realized in $\mathfrak{gl}(\infty)$ as follows:

$$e_i = \sum_{j\in r\mathbb{Z}+i} \psi(j-1)\psi^*(j), \quad f_i = \sum_{j\in r\mathbb{Z}+i} \psi(j)\psi^*(j-1),$$

$$\alpha_i^\vee = \sum_{j\in r\mathbb{Z}+i} (:\psi(j-1)\psi^*(j-1): - :\psi(j)\psi^*(j):) + \delta_{i0} \ \ (0 \le i \le r-1),$$

where : : denotes the normal ordering defined by

$$:\psi(i)\psi^*(j): = \psi(i)\psi^*(j) - \langle 0|\psi(i)\psi^*(j)|0\rangle.$$

The vacuum $|0\rangle$ generates an $A^{(1)}_{r-1}$-module in $\mathcal{F}[0]$ by the action of the Chevalley generators given above. It is shown in [8] that this is the basic $A^{(1)}_{r-1}$-module. Applying the boson-fermion correspondence to this case, we see that the representation space turns out to be $P^{(r)}$.

Let $\delta = \sum_{i=0}^{r-1} \alpha_i$ be the fundamental imaginary root. If we denote by W the Weyl group of type $A^{(1)}_{r-1}$, then the set of weights of the basic $A^{(1)}_{r-1}$-module is given by

$$P = \{w\Lambda_0 - n\delta \mid w \in W, \, n \in \mathbb{N}\}.$$

It is known that the maximal weight vectors are the Schur functions indexed by the r-core Young diagrams λ_c. We denote by $\Lambda(\lambda_c)$ the corresponding maximal weight. Remark that these Schur functions coincide with the r-reduced Schur functions. Then we have the following theorem.

Theorem 4.1 *The set of the r-reduced Schur functions*

$$\left\{ S^{(r)}_\lambda(t) \mid \lambda = (\lambda_c, \emptyset, \lambda[1], \ldots, \lambda[r-1]), \ \sum_{k=1}^{r-1} |\lambda[k]| = n \right\}$$

gives a basis for a weight space of the basic $A^{(1)}_{r-1}$-module with a weight $\Lambda(\lambda_c) - n\delta$.

Proof. A key for the proof is that to append a r-hook to λ is nothing but to let $\psi(m+r)\psi^*(m)$ act on Ψ for some $m \in \mathbb{Z}$.

We prove that $[\alpha_i^\vee, \psi(m+r)\psi^*(m)] = 0$ for $i = 0, \ldots, r-1$ and $m \in \mathbb{Z}$. However, this is a direct consequence of the identity

$$[\psi(k)\psi^*(k), \psi(m+r)\psi^*(m)] = \psi(k)\psi^*(m)\delta_{k,m+r} - \psi(m+r)\psi^*(k)\delta_{mk}.$$

On the other hand, d_0 acts on weighted homogeneous polynomials as the degree counting operator. Therefore, the r-reduced Schur function $S_\lambda^{(r)}(t)$, $\lambda = (\lambda_c, \emptyset, \lambda[1], \ldots, \lambda[r-1])$, is a weight vector of weight $\Lambda(\lambda_c) - n\delta$ if $\sum_{k=1}^{r-1} |\lambda[k]| = n$. We can show that these elements span the weight space by looking at the multiplicity formula

$$\sum_{n=0}^{\infty} \mathrm{mult}(\Lambda(\lambda_c) - n\delta) q^n = \frac{1}{\phi(q)^{r-1}}. \quad \blacksquare$$

§5 A Formula for Modular Characters

The formula (Theorem 3.1) can be interpreted as a formula of the Brauer characters of symmetric groups when r is a prime number. Let $(K, \mathcal{O}, k)$ be an r-modular system. In accordance with the decomposition of the identity into the sum of primitive central idempotents, the group algebra $k\mathfrak{S}_n$ decomposes into the direct sum of blocks, and the category of $k\mathfrak{S}_n$-modules decomposes into the direct sum of the module categories of these blocks. By the general theory on Specht modules S^λ, it is well known [4, 10] that

(1) Reduction modulo r of any $\mathcal{O}\mathfrak{S}_n$-submodule of an irreducible $k\mathfrak{S}_n$-module S^λ ($|\lambda| = n$) has the same set of composition factors all belonging to a single block.

(2) Blocks are parametrized by r-cores and S^λ belongs to the block corresponding to the r-core λ_c of λ.

(3) The Grothendieck group of the module category of the block corresponding to κ is generated by $\{[S^\lambda] \mid \lambda = (\lambda_c; \lambda[0], \ldots, \lambda[r-1])\}$ as a $\mathbb{Z}$-module.

(4) Let χ^λ be the ordinary character of S^λ. We consider it as a class function on r-regular classes. Then the Grothendieck group is isomorphic to the $\mathbb{Z}$-submodule of the space of class functions on r-regular classes spanned by $\{\chi^\lambda \mid \lambda = (\lambda_c; \lambda[0], \ldots, \lambda[r-1])\}$, in which $[S^\lambda]$ corresponds to χ^λ.

Now we can state a theorem. Because of the formula $S_\lambda = \sum_\mu (1/z_\mu)\chi^\lambda(\mu)p_\mu$, our formula (Theorem 3.1) is nothing but an equation for class functions on r-regular classes. Hence we have

Theorem 5.1 *(1)* $\{\chi^{(\lambda_c;\emptyset,\lambda[1],\ldots,\lambda[r-1])} \mid |\lambda_c| + r\sum_{k=1}^{r-1}|\lambda[k]| = n\}$ *is a basic set.*

(2) The following equation holds on r-regular classes:

$$\chi_\lambda^{(r)} = (-1)^{|\lambda[0]|}\delta_r(\lambda)\sum_{\mu,\nu_1,\ldots,\nu_{r-1}} LR^{\lambda[0]'}_{\nu_1\cdots\nu_{r-1}} LR^{\mu[1]}_{\nu_1\lambda[1]} \cdots LR^{\mu[r-1]}_{\nu_{r-1}\lambda[r-1]}\delta_r(\mu)\chi_\mu^{(r)},$$

where summation runs over Young diagram μ and $\nu_1, \ldots, \nu_{r-1}$ such that $|\mu| = |\lambda|$, $\mu[0] = \emptyset$ and the core of μ coincides with that of λ.

As is pointed out by A. Gyoja [3], we can lift this situation to the Hecke algebra $H_n(q)$ with $q = \zeta_r$, a primitive r-th root of unity. The K-algebra $H_n(\zeta_r)$ is defined by generators $T_1, \ldots, T_{n-1}$ and relations

$$(T_i + 1)(T_i - \zeta_r) = 0 \;\; (1 \le i \le n-1),$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \;\; (1 \le i \le n-2),$$

$$T_iT_j = T_jT_i \;\; (|i-j| \ge 2).$$

The reduction modulo r of the $\mathcal{O}$-lattice $\sum_{w\in\mathfrak{S}_n}\mathcal{O}T_w$ is isomorphic to $k\mathfrak{S}_n$. Hence we have a canonical $\mathbb{Z}$-linear map ϕ from the Grothendieck group of $H_n(\zeta_r)$-modules to that of $k\mathfrak{S}_n$-modules. By [2, Theorem 7.7], the decomposition matrices of both are lower unitriangular with respect to the same ordering of partitions. Hence we have

$$\begin{bmatrix} 1 & & 0 \\ * & \ddots & \\ & * & 1 \end{bmatrix}\begin{bmatrix} [D_{\lambda_1}] \\ [D_{\lambda_2}] \\ \vdots \\ [D_{\lambda_l}] \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ * & \ddots & \\ & * & 1 \end{bmatrix}\begin{bmatrix} [E_{\lambda_1}] \\ [E_{\lambda_2}] \\ \vdots \\ [E_{\lambda_l}] \end{bmatrix},$$

where $\{D_{\lambda_1},\dots,D_{\lambda_l}\}$ are irreducible $k\mathfrak{S}_n$-modules and $\{E_{\lambda_1},\dots,E_{\lambda_l}\}$ are irreducible $H_n(\zeta_r)$-modules. By multiplying $\left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1 & \end{array}\right]$ to the both sides, we have

$$\begin{bmatrix} [D_{\lambda_1}] \\ \vdots \\ [D_{\lambda_l}] \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ * & \ddots & \\ & & 1 \end{bmatrix}\begin{bmatrix} [E_{\lambda_1}] \\ \vdots \\ [E_{\lambda_l}] \end{bmatrix}.$$

Therefore ϕ is invertible and hence these Grothendieck groups are isomorphic.

Corollary 5.2

$$[S_\lambda^{(r)}] = (-1)^{|\lambda[0]|}\delta_r(\lambda)\sum_{\mu,\nu_1,\dots,\nu_{r-1}} LR^{\lambda[0]'}_{\nu_1\cdots\nu_{r-1}} LR^{\mu[1]}_{\nu_1\lambda[1]}\cdots LR^{\mu[r-1]}_{\nu_{r-1}\lambda[r-1]}\delta_r(\mu)[S_\mu^{(r)}],$$

where S^λ is the Specht module for the Hecke algebra $H_n(\zeta_r)$.

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